

When are two Dedekind sums equal?

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Abstract

A natural question about Dedekind sums is to find conditions on the integers a_1, a_2 , and b such that $s(a_1, b) = s(a_2, b)$. We prove that if the former equality holds then $b \mid (a_1 a_2 - 1)(a_1 - a_2)$. Surprisingly, to the best of our knowledge such statements have not appeared in the literature. A similar theorem is proved for the more general Dedekind-Rademacher sums as well, namely that for any fixed non-negative integer n , a positive integer modulus b , and two integers a_1 and a_2 that are relatively prime to b , the hypothesis $r_n(a_1, b) = r_n(a_2, b)$ implies that $b \mid (6n^2 + 1 - a_1 a_2)(a_2 - a_1)$.

1 Introduction

Dedekind sums arise naturally in many fields, most prominently in combinatorial geometry [1] and in the theory of modular forms [2]. The classical **Dedekind sum** is defined by:

$$s(a, b) = \sum_{k=0}^{b-1} \left(\left(\frac{ka}{b} \right) \right) \left(\left(\frac{k}{b} \right) \right),$$

where a and b are any two relatively prime integers, and where the **Sawtooth function** is defined by

$$((x)) = \begin{cases} \{x\} - \frac{1}{2} & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

The Dedekind sum enjoys two important properties. The first of these properties is the periodicity of the Dedekind sums in the first variable, namely $s(a + kb, b) = s(a, b)$ for all $k \in \mathbb{Z}$. The second, and deeper, of these properties is the famous reciprocity law for Dedekind sums:

$$s(a, b) + s(b, a) = -\frac{1}{4} + \frac{1}{12} \left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a} \right),$$

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valid for any two relatively prime integers a and b . It is very natural to ask under what conditions on the integers a_1, a_2 , and b is it true that

$$s(a_1, b) = s(a_2, b)?$$

This question also arises in topological considerations, and involves the correction terms of the Heegaard Floer Homology [3]. We answer this question with the following results.

Theorem 1.1. *Let b be a positive integer, and a_1, a_2 any two integers that are relatively prime to b . If $s(a_1, b) = s(a_2, b)$, then*

$$b \mid (1 - a_1 a_2)(a_1 - a_2).$$

An immediate corollary of this theorem is the following result:

Corollary 1.2. *Let p be a prime. Then $s(a_1, p) = s(a_2, p)$ if and only if $a_1 \equiv a_2 \pmod{p}$, or $a_1 a_2 \equiv 1 \pmod{p}$.*

We note that the converse of Theorem 1.1 is false in general. Consider, for example, $b = 40$, and $a_1 = 37, a_2 = 33$. Then $b \mid (1 - 37 \cdot 33)(37 - 33) = 20 \cdot 4$, and yet $s(37, 40) = -\frac{13}{16}$ and $s(33, 40) = -\frac{5}{16}$, so that $s(a_1, b) \neq s(a_2, b)$ in this case.

We also study the analogous question for the Dedekind-Rademacher sums, which arise in Donald Knuth's work on pseudo-random number generators. Given any non-negative integer n , and any two relatively prime integers a and b , we define the Dedekind-Rademacher sum by:

$$r_n(a, b) = \sum_{k=0}^{b-1} \left(\left(\frac{ka + n}{b} \right) \right) \left(\left(\frac{k}{b} \right) \right).$$

In order to state the corresponding reciprocity law for the Dedekind-Rademacher sums, we define

$$\chi_a(n) = \begin{cases} 1 & \text{if } a \mid n, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1.3 (Reciprocity law for Dedekind-Rademacher sums). *Let a and b be relatively prime positive integers. Then for $n = 1, 2, \dots, a + b$,*

$$\begin{aligned} r_n(a, b) + r_n(b, a) &= \frac{n^2}{2ab} - \frac{n}{2} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{ab} \right) + \frac{1}{12} \left(\frac{b}{a} + \frac{a}{b} + \frac{1}{ab} \right) \\ &\quad + \frac{1}{2} \left(\left(\left(\frac{a^{-1}n}{b} \right) \right) + \left(\left(\frac{b^{-1}n}{a} \right) \right) + \left(\left(\frac{n}{a} \right) \right) + \left(\left(\frac{n}{b} \right) \right) \right) \\ &\quad + \frac{1}{4} (1 + \chi_a(n) + \chi_b(n)), \end{aligned}$$

where $aa^{-1} \equiv 1 \pmod{b}$ and $bb^{-1} \equiv 1 \pmod{a}$.

The proof of Lemma 1.3 can be found, for example, in [1]. For the Dedekind-Rademacher sums, we have the following two results.

Theorem 1.4. Fix a non-negative integer n and a positive integer b . Let a_1 and a_2 be any two integers that are relatively prime to b .

If $r_n(a_1, b) = r_n(a_2, b)$, then

$$b \mid (6n^2 + 1 - a_1 a_2)(a_2 - a_1).$$

Again, an immediate Corollary for prime moduli follows.

Corollary 1.5. Let p be a prime. If $r_n(a_1, p) = r_n(a_2, p)$, then it follows that $a_1 \equiv a_2 \pmod{p}$, or

$$a_1 a_2 \equiv 1 + 6n^2 \pmod{p}.$$

Note that the converses of Theorem 1.4 and Corollary 1.5 are generally false. A counter example is provided by $a_1 = 3$, $a_2 = 11$, $b = 23$ and $n = 6$ for which $b \mid (6n^2 + 1 - a_1 a_2)$ but $r_6(3, 23) = -\frac{3}{92}$ while $r_6(11, 23) = \frac{43}{92}$.

As a direction for further research, we note that when b is composite and c is rational, the equation $s(x, b) = c$ might have more than 2 solutions in $x \in \mathbb{Z}$. In fact, Corollary 1.2 shows that if b has r distinct prime divisors, then the number of solutions to $s(x, b) = c$ is greater than or equal to 2^r , by the usual elementary modular arithmetic arguments. It would be quite interesting to study how many integer solutions in $x \in \mathbb{Z}$ the equation $s(x, b) = c$ has in general.

2 Proofs

We first introduce some lesser-known but useful properties of Dedekind sums. It is proved in [2] that

$$6b s(a, b) \in \mathbb{Z}, \tag{2.1}$$

for any two relatively prime integer a and b . This property of Dedekind sums gives us a nice upper bound on the denominators that any Dedekind sum $s(a, b)$ may have, and it plays an interesting role in the proof of Theorem 1.1.

Proof of Theorem 1.1. For any integers a_1 relatively prime to b , and a_2 relatively prime to b , Dedekind's Reciprocity law implies that we have the following two identities:

$$12a_1 b (s(a_1, b) + s(b, a_1)) = -3a_1 b + a_1^2 + b^2 + 1, \tag{2.2}$$

$$12a_2 b (s(a_2, b) + s(b, a_2)) = -3a_2 b + a_2^2 + b^2 + 1. \tag{2.3}$$

Multiplying (2.2) with a_2 , and multiplying (2.3) with a_1 , we get

$$12a_1 a_2 b (s(a_1, b) + s(b, a_1)) = a_2 (-3a_1 b + a_1^2 + b^2 + 1), \tag{2.4}$$

$$12a_1 a_2 b (s(a_2, b) + s(b, a_2)) = a_1 (-3a_2 b + a_2^2 + b^2 + 1). \tag{2.5}$$

Subtracting (2.5) from (2.4) gives us

$$\begin{aligned} & 12a_1a_2b(s(a_1, b) + s(b, a_1)) - 12a_1a_2b(s(a_2, b) + s(b, a_2)) \\ &= a_2(-3a_1b + a_1^2 + b^2 + 1) - a_1(-3a_2b + a_2^2 + b^2 + 1). \end{aligned} \quad (2.6)$$

We know, by assumption, that $s(a_1, b) = s(a_2, b)$, and therefore

$$12a_1a_2bs(b, a_1) - 12a_1a_2bs(b, a_2) = a_1^2a_2 + b^2a_2 - b^2a_1 + a_2 - a_2^2a_1 - a_1. \quad (2.7)$$

Using the fact (2.1) that $(6a_1)s(b, a_1)$ and $(6a_2)s(b, a_2)$ are both integers, we may reduce (2.7) mod b to obtain the result

$$(a_2 - a_1)(1 - a_1a_2) \equiv 0 \pmod{b}. \quad (2.8)$$

□

Lemma 2.1. *For any relatively prime integers a and b , we have*

$$12b r_n(a, b) \in \mathbb{Z}.$$

Proof. Note that for relatively prime numbers a and b , there's always a solution to the equation $ka + n \equiv 0 \pmod{b}$, while $k \in \{0, 1, \dots, b-1\}$. We consider two different situations.

- When $k = 0$, or equivalently if $n \equiv 0 \pmod{b}$, then $r_n(a, b) = r_0(a, b) = s(a, b)$ and it was pointed out in [2] that $6bs(a, b) \in \mathbb{Z}$.
- When $k = k_0a + n \equiv 0 \pmod{b}$ where $k_0 \in \{1, \dots, b-1\}$.

$$\begin{aligned} 12br_n(a, b) &= 12b \sum_{k=0}^{b-1} \left(\left(\frac{ka+n}{b} \right) \right) \left(\left(\frac{k}{b} \right) \right), \\ &= 12b \sum_{k=1, k \neq k_0}^{b-1} \left(\frac{ka+n}{b} - \left[\frac{ka+n}{b} \right] - \frac{1}{2} \right) \left(\frac{k}{b} - \frac{1}{2} \right), \\ &= 12b \sum_{k=1, k \neq k_0}^{b-1} \left(\frac{k(ka+n)}{b^2} - \frac{A}{2b} + \frac{1}{4} \right), \\ &= 12b \left(\frac{a(b-1)(2b-1)}{6b} + \frac{n(b-1)}{2b} - \frac{A(b-2)}{2b} + \frac{b-2}{4} - \frac{Ck_0}{b} \right), \\ &= 2a(b-1)(2b-1) + 6n(b-1) - 6A(b-2) + 3b(b-2) - 12Ck_0. \end{aligned}$$

where $A, C \in \mathbb{Z}$, and immediately we have $12br_n(a, b) \in \mathbb{Z}$.

□

Proof of Theorem 1.4. From the reciprocity law for the Dedekind-Rademacher sums, we know that when a_1, a_2 are relatively prime to b , we have:

$$\begin{aligned} 12a_1b(r_n(a_1, b) + r_n(b, a_1)) &= 6n^2 + a_1^2 + b^2 + 1 - 9a_1b \\ &\quad - 6a_1b \left(\left[\frac{a_1^{-1}n}{b} \right] + \left[\frac{b^{-1}n}{a_1} \right] + \left[\frac{n}{b} \right] + \left[\frac{n}{a_1} \right] \right) \\ &\quad + 3a_1b(\chi_{a_1}(n) + \chi_b(n)), \end{aligned} \quad (2.9)$$

$$\begin{aligned} 12a_2b(r_n(a_2, b) + r_n(b, a_2)) &= 6n^2 + a_2^2 + b^2 + 1 - 9a_2b \\ &\quad - 6a_2b \left(\left[\frac{a_2^{-1}n}{b} \right] + \left[\frac{b^{-1}n}{a_2} \right] + \left[\frac{n}{b} \right] + \left[\frac{n}{a_2} \right] \right) \\ &\quad + 3a_2b(\chi_{a_2}(n) + \chi_b(n)). \end{aligned} \quad (2.10)$$

To simplify the ensuing algebra, we let

$$\begin{aligned} S_{a_1} &= \left[\frac{a_1^{-1}n}{b} \right] + \left[\frac{b^{-1}n}{a_1} \right] + \left[\frac{n}{b} \right] + \left[\frac{n}{a_1} \right] \in \mathbb{Z}, \\ S_{a_2} &= \left[\frac{a_2^{-1}n}{b} \right] + \left[\frac{b^{-1}n}{a_2} \right] + \left[\frac{n}{b} \right] + \left[\frac{n}{a_2} \right] \in \mathbb{Z}, \\ T_{a_1} &= \chi_{a_1}(n) + \chi_b(n) \in \mathbb{Z}, \\ T_{a_2} &= \chi_{a_2}(n) + \chi_b(n) \in \mathbb{Z}, \end{aligned}$$

we can rewrite (2.9) and (2.10) as follows:

$$12a_1b(r_n(a_1, b) + r_n(b, a_1)) = 6n^2 + a_1^2 + b^2 + 1 - 9a_1b - 6a_1bS_{a_1} + 3a_1bT_{a_1}, \quad (2.11)$$

$$12a_2b(r_n(a_2, b) + r_n(b, a_2)) = 6n^2 + a_2^2 + b^2 + 1 - 9a_2b - 6a_2bS_{a_2} + 3a_2bT_{a_2}. \quad (2.12)$$

Multiplying (2.11) with a_2 gives us

$$\begin{aligned} 12a_1a_2b(r_n(a_1, b) + r_n(b, a_1)) \\ = a_2(6n^2 + a_1^2 + b^2 + 1 - 9a_1b - 6a_1bS_{a_1} + 3a_1bT_{a_1}). \end{aligned} \quad (2.13)$$

Multiplying (2.12) with a_1 gives us

$$\begin{aligned} 12a_1a_2b(r_n(a_2, b) + r_n(b, a_2)) \\ = a_1(6n^2 + a_2^2 + b^2 + 1 - 9a_2b - 6a_2bS_{a_2} + 3a_2bT_{a_2}). \end{aligned} \quad (2.14)$$

Since $r_n(a_1, b) = r_n(a_2, b)$, subtracting (2.14) from (2.13) we get

$$\begin{aligned} 12a_1a_2b(r_n(b, a_1) - r_n(b, a_2)) \\ = (a_2 - a_1)(6n^2 + 1 - a_1a_2) + b^2(a_2 - a_1) \\ - 6a_1a_2b(S_{a_1} + S_{a_2}) + 3a_1a_2b(T_{a_1} + T_{a_2}). \end{aligned} \quad (2.15)$$

We notice that, by Lemma (2.1), we have $12a_1r_n(b, a_1) \in \mathbb{Z}$ and $12a_2r_n(b, a_2) \in \mathbb{Z}$. We may therefore reduce both sides of (2.15) modulo b to obtain the result:

$$0 \equiv (6n^2 + 1 - a_1a_2)(a_2 - a_1) \pmod{b}.$$

□

References

- [1] MATTHIAS BECK, SINAI ROBINS, *Computing the continuous discretely: integer point enumeration in polyhedra*, Springer, 1st Edition, 2007.
- [2] HANS RADEMACHER, EMIL GROSSWALD, *Dedekind sums*, The Carus Mathematical Monographs, The Mathematical Association of America, 1st Edition, 1972.
- [3] BOYER, STEVEN, LINES, DANIEL, *Surgery formulae for Casson's invariant and extensions to homology lens spaces*, J. Reine Angew. Math, Vol 405, pp. 181- 220,1990.
- [4] R.C.GRIMSON, *Reciprocity Theorem For Dedekind Sums*, The American Mathematical Monthly, Vol 81, pp. 747-749, 1974.

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